

# Quantum string cosmology in the phase space

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## Abstract

Deformation quantization is applied to quantize gravitational systems coupled with matter. This quantization procedure is performed explicitly for quantum cosmology of these systems in a flat minisuper(phase)space. The procedure is employed in a quantum string minisuperspace corresponding to an axion-dilaton system in an isotropic FRW Universe. The Wheeler-DeWitt-Moyal equation is obtained and its corresponding Wigner function is given analytically in terms of Meijer's functions. Finally, this Wigner functions is used to extract physical information of the system.

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## I. INTRODUCTION

Our contribution to the *VIII Mexican workshop on gravitation and mathematical-physics 2010* focused on explaining the implementation of the deformation quantization formalism to a gravitational field coupled to matter. Also it was considered its application to diverse cosmological models and the baby Universe system. The analysis included a minisuperspace approach of the de Sitter model, the Kantowski-Sachs Universe (for the commutative and non-commutative cases) and finally we discussed the case of string cosmology with a dilaton exponential potential. The detailed construction is presented in [1] (and references therein) and will not be repeated here. Instead we will apply the formalism to another case using the minisuperspace approach of string cosmological models involving axion and dilaton fields in a curved space. The issues we will discuss in section two involve mainly a brief review of the Weyl-Wigner-Groenewold-Moyal (WWGM) formalism of deformation quantization [2] in the context of quantum cosmological models. This is an integral functional formalism and it is well defined in the case of the whole Wheeler's superspace of 3-metrics in the ADM's description of gravity and matter.

The four dimensional low energy effective field theory action of string theory contains at least three massless fields: the graviton, the axion and the dilaton [3]. Indirect evidences of low energy string theory can appear in the physical consequences of the axion in a curved spacetime [4]. In string theory the dilaton is quite relevant since it defines the string coupling constant.

One of the most important models of string cosmology is the pre-big-bang scenario [5]. It incorporates the target space string duality through the *scale factor duality*. This predicts a decreasing curvature for negative values of the time coordinate. This curvature is the specular image of the so-called *post-big-bang* cosmology. The pre-big-bang cosmology scenario is important as it is a modification produced by string theory which is (at least perturbatively) a consistent theory of quantum gravity. Thus it is expected to describe the correct modification to standard general relativity at very early times. The prediction is that there is not a standard big-bang singularity since it is smoothed by the scale factor duality coming from target space duality. In the present paper we study precisely the quantum cosmological model regarding the pre-big-bang scenario in the context of the quantization of the phase space of the minisuperspace of these models. In particular we discuss the cosmological model

of a string theory in four dimensions with axion and dilaton fields in a curved space [6].

The rest of the paper is organized as follows. In section three, we apply the deformation quantization formalism to string cosmology with axion and dilation in a curved minisuperspace approach and we find the exact Wigner functional which is equivalent to the quantum states of the Universe. In the last section, we give our final remarks.

## II. DEFORMATION QUANTIZATION OF THE GRAVITATIONAL FIELD COUPLED TO MATTER

We start by giving a brief overview of the Hamiltonian formalism for the gravitational field. In particular we consider the ADM decomposition of general relativity coupled to matter (see for instance [7]). We take a globally hyperbolic spacetime modeled with a pseudo Riemannian manifold  $(M, g)$  where  $M = \Sigma \times R$  and  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - N^i N_i)(dt)^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j$ , with signature  $(-, +, +, +)$ . Here  $h_{ij}$  is the intrinsic metric on the hypersurface  $\Sigma$ ,  $N$  is the lapse function and  $N^i$  is the shift vector. The *superspace* is defined by  $\text{Riem}(\Sigma) = \{h_{ij}(x), \Phi(x) | x \in \Sigma\}$  where  $\Phi$  is the scalar field. Let  $\text{Met}(\Sigma) = \{h_{ij}(x) | x \in \Sigma\}$  which is an infinite dimensional manifold. The moduli space of the theory is  $\mathcal{M} = \frac{\text{Riem}(\Sigma)}{\text{Diff}(\Sigma)}$  or for pure gravity  $\mathcal{M} = \frac{\text{Met}(\Sigma)}{\text{Diff}(\Sigma)}$  where  $\text{Diff}(\Sigma)$  is the group of diffeomorphisms of  $\Sigma$ . The corresponding phase space is given by  $\Gamma^* \cong T^*\text{Met}(\Sigma) = \{(h_{ij}(x), \pi^{ij}(x))\}$ , where  $\pi^{ij} = \frac{\partial L_{EH}}{\partial h_{ij}}$  and  $L_{EH}$  is the Einstein-Hilbert Lagrangian. In the following we will deal with fields at the moment  $t = 0$  (on  $\Sigma$ ) and we put  $h_{ij}(x, 0) \equiv h_{ij}(x)$  and  $\pi^{ij}(x, 0) \equiv \pi^{ij}(x)$ .

The kind of systems considered here are invariant under diffeomorphisms thus their canonical Hamiltonian is pure constraint. Therefore we only should worry to solve the constraints of the theory. These are the momentum and the Hamiltonian constraint. This latter is written as

$$\mathcal{H}_\perp(x) = 4\vartheta^2 G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{h}}{4\vartheta^2} ({}^3R - 2\Lambda) + \frac{1}{2} \sqrt{h} \left( \frac{\pi^2}{h} + h^{ij} \Phi_{,i} \Phi_{,j} + 2V(\Phi) \right) = 0, \quad (1)$$

where  $\vartheta^2 = 4\pi G_N$ ,  $\pi = \frac{\partial L_M}{\partial \Phi}$ , with  $L_M$  stands for the matter Lagrangian,  $\Lambda$  is the cosmological constant,  ${}^3R(h)$  is the scalar curvature of  $\Sigma$ ,  $G_{ijkl} = \frac{1}{2} h^{-1/2} (h_{ik} h_{jl} + h_{il} h_{jk} - h_{ij} h_{kl})$ ,  $_{,j}$  denotes partial derivative with respect to  $x^j$  and  $h$  is the determinant of  $h_{ij}$ . One of the most important structures for quantization is the Poisson bracket between  $h_{ij}$  and  $\pi^{kl}$  given

by

$$\{h_{ij}(x), \pi^{kl}(y)\}_{PB} = \frac{1}{2}(\delta_i^k \delta_j^l + \delta_j^k \delta_i^l) \delta(x - y). \quad (2)$$

The canonical quantization promotes the canonical variables to operators acting on some Hilbert space (or Fock space). In the  $\hbar$ -representation they look like:  $\hat{h}_{ij}|h_{ij}, \Phi\rangle = h_{ij}|h_{ij}, \Phi\rangle$ ,  $\hat{\pi}^{ij}|h_{ij}, \Phi\rangle = -i\hbar \frac{\delta}{\delta h_{ij}(x)}|h_{ij}, \Phi\rangle$ ,  $\hat{\Phi}|h_{ij}, \Phi\rangle = \Phi(x)|h_{ij}, \Phi\rangle$  and  $\hat{\pi}_\Phi|h_{ij}, \Phi\rangle = -i\hbar \frac{\delta}{\delta \Phi(x)}|h_{ij}, \Phi\rangle$ . These operators satisfy the commutation relations

$$[\hat{h}_{ij}(x), \hat{\pi}^{kl}(y)] = \frac{i\hbar}{2}(\delta_i^k \delta_j^l + \delta_j^k \delta_i^l) \delta(x - y), \quad (3)$$

and similar expressions for the matter part. The Hamiltonian constraint at the quantum level is given by  $\hat{\mathcal{H}}_\perp|\Psi\rangle = 0$ . In the  $\hbar$ -representation we have the constraint given by

$$\left[ -4\vartheta^2 G_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} + \frac{\sqrt{\hbar}}{4\vartheta^2} \left( -{}^3R(h) + 2\Lambda + 4\kappa^2 \hat{T}^{00} \right) \right] \Psi[h_{ij}, \Phi] = 0, \quad (4)$$

where  $\langle h_{ij}, \Phi | \Psi \rangle = \Psi[h_{ij}, \Phi]$  and  $\hat{T}^{00} = -\frac{1}{2\hbar} \frac{\delta^2}{\delta \Phi^2} + \frac{1}{2} h^{ij} \Phi_{,i} \Phi_{,j} + V(\Phi)$ . This equation is called the Wheeler-DeWitt (WDW) equation for the wave function of the Universe  $\Psi[h_{ij}, \Phi]$ .

The deformation quantization of gravity in ADM formalism and constrained systems has been considered in [8]. In the rest of the paper we assume that the superspace has a flat metric  $G_{ijkl}$  to ensure the existence of the Fourier transform.

Let  $F[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi]$  be a functional on the phase space  $\Gamma^*$  (Wheeler's phase superspace) and let  $\tilde{F}[\mu^{ij}, \lambda_{ij}; \mu, \lambda]$  be its Fourier transform

$$\begin{aligned} \tilde{F}[\mu^{ij}, \lambda_{ij}; \mu, \lambda] = \int \mathcal{D}\pi^{ij} \mathcal{D}h_{ij} \mathcal{D}\pi_\Phi \mathcal{D}\Phi \exp \left\{ -i \int dx \left( \mu^{ij}(x) h_{ij}(x) + \lambda_{ij}(x) \pi^{ij}(x) \right. \right. \\ \left. \left. + \mu(x) \Phi(x) + \lambda(x) \pi_\Phi(x) \right) \right\} F[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi], \end{aligned} \quad (5)$$

where  $\mathcal{D}h_{ij} = \prod_x dh_{ij}(x)$ ,  $\mathcal{D}\pi^{ij} = \prod_x d\pi^{ij}(x)$ ,  $\mathcal{D}\Phi = \prod_x d\Phi(x)$ ,  $\mathcal{D}\pi_\Phi = \prod_x d\pi_\Phi(x)$ . By analogy to the quantum mechanics case, we define the Weyl quantization map  $\mathcal{W}$  as follows

$$\hat{F} = \mathcal{W}(F[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi]) := \int \mathcal{D} \left( \frac{\lambda_{ij}}{2\pi} \right) \mathcal{D} \left( \frac{\mu^{ij}}{2\pi} \right) \mathcal{D} \left( \frac{\lambda}{2\pi} \right) \mathcal{D} \left( \frac{\mu}{2\pi} \right) \tilde{F}[\mu^{ij}, \lambda_{ij}; \mu, \lambda] \hat{\mathcal{U}}[\mu^{ij}, \lambda_{ij}; \mu, \lambda], \quad (6)$$

where

$$\hat{\mathcal{U}}[\mu^{ij}, \lambda_{ij}; \mu, \lambda] := \exp \left\{ i \int dx \left( \mu^{ij}(x) \hat{h}_{ij}(x) + \lambda_{ij}(x) \hat{\pi}^{ij}(x) + \mu(x) \hat{\Phi}(x) + \lambda(x) \hat{\pi}_\Phi(x) \right) \right\}. \quad (7)$$

Here  $\hat{h}_{ij}$ ,  $\hat{\pi}^{ij}$ ,  $\hat{\Phi}$  and  $\hat{\pi}_\Phi$  are the field operators defined by:  $\hat{h}_{ij}(x)|h_{ij}, \Phi\rangle = h_{ij}(x)|h_{ij}, \Phi\rangle$ ,  $\hat{\pi}^{ij}(x)|\pi_{ij}, \pi_\Phi\rangle = \pi_{ij}(x)|\pi_{ij}, \pi_\Phi\rangle$ ,  $\hat{\Phi}(x)|h_{ij}, \Phi\rangle = \Phi(x)|h_{ij}, \Phi\rangle$ ,  $\hat{\pi}_\Phi(x)|\pi_{ij}, \pi_\Phi\rangle = \pi_\Phi(x)|\pi_{ij}, \pi_\Phi\rangle$ .

The Campbell-Baker-Hausdorff formula, commutator algebra and the completeness relations lead to an explicit form for the operator  $\widehat{\mathcal{U}}$  to be

$$\begin{aligned} \widehat{\mathcal{U}}[\mu^{ij}, \lambda_{ij}; \mu, \lambda] &= \int \mathcal{D}h_{ij} \mathcal{D}\Phi \exp \left\{ i \int dx \mu^{ij}(x) h_{ij}(x) + \mu(x) \Phi(x) \right\} \\ &\times \left| h_{ij} - \frac{\hbar \lambda_{ij}}{2}, \Phi - \frac{\hbar \lambda}{2} \right\rangle \left\langle h_{ij} + \frac{\hbar \lambda_{ij}}{2}, \Phi + \frac{\hbar \lambda}{2} \right|. \end{aligned} \quad (8)$$

One immediately obtains the following structure

$$\widehat{F} = \int \mathcal{D} \left( \frac{\pi^{ij}}{2\pi\hbar} \right) \mathcal{D}h_{ij} \mathcal{D} \left( \frac{\pi_\Phi}{2\pi\hbar} \right) \mathcal{D}\Phi F[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] \widehat{\Omega}[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi], \quad (9)$$

where the operator  $\widehat{\Omega}$  is the Stratonovich-Weyl quantizer and it is given by

$$\begin{aligned} \widehat{\Omega}[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] &= \int \mathcal{D} \left( \frac{\hbar \lambda_{ij}}{2\pi} \right) \mathcal{D}\mu^{ij} \mathcal{D} \left( \frac{\hbar \lambda}{2\pi} \right) \mathcal{D}\mu \\ &\times \exp \left\{ -i \int dx \left( \mu^{ij}(x) h_{ij}(x) + \lambda_{ij}(x) \pi^{ij}(x) + \mu(x) \Phi(x) + \lambda(x) \pi_\Phi(x) \right) \right\} \widehat{\mathcal{U}}[\mu^{ij}, \lambda_{ij}; \mu, \lambda]. \end{aligned} \quad (10)$$

This operator can be written in the following form (that can be very useful to invert the mapping  $\mathcal{W}$ )

$$\begin{aligned} \widehat{\Omega}[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] &= \int \mathcal{D}\xi_{ij} \int \mathcal{D}\xi \exp \left\{ -\frac{i}{\hbar} \int dx \xi_{ij}(x) \pi^{ij}(x) + \xi(x) \pi_\Phi(x) \right\} \\ &\times \left| h_{ij} - \frac{\xi_{ij}}{2}, \Phi - \frac{\xi}{2} \right\rangle \left\langle h_{ij} + \frac{\xi_{ij}}{2}, \Phi + \frac{\xi}{2} \right|. \end{aligned} \quad (11)$$

The space  $\mathcal{A}$  of all functionals on the phase space  $\Gamma^*$  i.e.  $\mathcal{A} := \{F = F[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi]\}$ , forms with the usual product an associative and commutative algebra. This algebra can be deformed into an associative and non-commutative algebra  $\mathcal{A}^\star$  with the  $\star$ -product. In relation to the  $\mathcal{W}$  map this  $\star$  product is defined as:  $\mathcal{W}^{-1}(\widehat{F} \cdot \widehat{G}) = F \star G$  for any pair of functionals  $F$  and  $G$  in  $\mathcal{A}$ . Thus the  $\star$  product is defined as

$$(F \star G)[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] := \mathcal{W}^{-1}(\widehat{F} \widehat{G}) = \text{Tr} \left\{ \widehat{\Omega}[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] \widehat{F} \widehat{G} \right\}, \quad (12)$$

or after some straightforward computations

$$(F \star G)[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] = F[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] \exp \left\{ \frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}} \right\} G[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi], \quad (13)$$

where  $\overleftrightarrow{\mathcal{P}}$  is the operator given by

$$\overleftrightarrow{\mathcal{P}} := \int dx \left( \frac{\overleftarrow{\delta}}{\delta h_{ij}(x)} \frac{\overrightarrow{\delta}}{\delta \pi^{ij}(x)} - \frac{\overleftarrow{\delta}}{\delta \pi^{ij}(x)} \frac{\overrightarrow{\delta}}{\delta h_{ij}(x)} \right) + \int dx \left( \frac{\overleftarrow{\delta}}{\delta \Phi(x)} \frac{\overrightarrow{\delta}}{\delta \pi_\Phi(x)} - \frac{\overleftarrow{\delta}}{\delta \pi_\Phi(x)} \frac{\overrightarrow{\delta}}{\delta \Phi(x)} \right). \quad (14)$$

Let  $\hat{\rho}$  be the density operator of a quantum state. The functional  $\rho_W[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi]$  corresponding to  $\hat{\rho}$  reads

$$\begin{aligned}\rho_W[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] &= \text{Tr} \left\{ \hat{\Omega}[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] \hat{\rho} \right\} \\ &:= \int \mathcal{D} \left( \frac{\xi_{ij}}{2\pi\hbar} \right) \mathcal{D} \left( \frac{\xi}{2\pi\hbar} \right) \exp \left\{ -\frac{i}{\hbar} \int dx \left( \xi_{ij}(x) \pi^{ij}(x) + \xi(x) \pi_\Phi(x) \right) \right\} \\ &\quad \times \left\langle h_{ij} + \frac{\xi_{ij}}{2}, \Phi + \frac{\xi}{2} \left| \hat{\rho} \right| h_{ij} - \frac{\xi_{ij}}{2}, \Phi - \frac{\xi}{2} \right\rangle.\end{aligned}\tag{15}$$

For a pure state of the system  $\hat{\rho} = |\Psi\rangle\langle\Psi|$  we have

$$\begin{aligned}\rho_W[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] &= \int \mathcal{D} \left( \frac{\xi_{ij}}{2\pi\hbar} \right) \mathcal{D} \left( \frac{\xi}{2\pi\hbar} \right) \exp \left\{ -\frac{i}{\hbar} \int dx \left( \xi_{ij}(x) \pi^{ij}(x) + \xi(x) \pi_\Phi(x) \right) \right\} \\ &\quad \times \Psi^* \left[ h_{ij} - \frac{\xi_{ij}}{2}, \Phi - \frac{\xi}{2} \right] \Psi \left[ h_{ij} + \frac{\xi_{ij}}{2}, \Phi + \frac{\xi}{2} \right].\end{aligned}\tag{16}$$

It is now possible to write the Hamiltonian constraint in terms of the  $\star$ -product and the Wigner functional as

$$\mathcal{H}_\perp \star \rho_W[h_{ij}, \pi^{ij}; \Phi, \pi_\Phi] = 0.\tag{17}$$

This is the Moyal deformation of the Wheeler-DeWitt equation and we will just call it the Wheeler-DeWitt-Moyal (WDWM) equation.

### III. DEFORMATION QUANTIZATION IN QUANTUM COSMOLOGY

String theory can be employed to describe the evolution of the early Universe and one of the most important areas of research are the cosmological consequences of the dilaton and its role in the pre-big-bang scenario [5]. An interesting case to deal with under the deformation quantization procedure is the axion-dilaton quantum cosmology in curved space [6].

#### A. Axion-dilaton quantum cosmology in curved space

Let's start with a model in a FRW metric with  $\Lambda = 0$ , axion energy density, dilaton field and spatial curvature described by the following effective action

$$S = \frac{\lambda_s}{2} \int d\tau \left[ -\bar{\phi}'^2 + \beta'^2 - e^{-2\bar{\phi}} \left( \frac{1}{2} q^2 e^{-2\sqrt{3}\beta} - 6\kappa e^{-2\beta/\sqrt{3}} \right) \right],\tag{18}$$

where  $\lambda_s$  denotes the string length,  $q^2$  codifies the contribution of axion energy density [6],  $\kappa$  is related to the spatial curvature and the prime variables stand for the derivatives with

respect to the *dilaton time* [9].

Using now the variables  $\phi = \bar{\phi} + \sqrt{3}\beta$ ,  $y = \bar{\phi}\sqrt{3} + \beta = \sqrt{3}\left(\phi - 2\ln a - \ln \int \frac{d^3x}{\lambda_s^3}\right)$ , we obtain that

$$S = \frac{\lambda_s}{4} \int d\tau \left[ \phi'^2 - y'^2 - q^2 e^{-2\phi} + 12\kappa e^{-2y/\sqrt{3}} \right]. \quad (19)$$

The corresponding WDW equation takes the following form

$$\hat{H}\Psi(y, \phi) = \frac{1}{\lambda_s} \left( \hbar^2 \partial_y^2 - \hbar^2 \partial_\phi^2 + \frac{1}{4} \lambda_s^2 q^2 e^{-2\phi} - 3\lambda_s^2 \kappa e^{-\frac{2}{\sqrt{3}}y} \right) \Psi(y, \phi) = 0. \quad (20)$$

The solutions of this equation are obtained by the method of separation of variables  $\Psi(y, \phi) = \chi_\alpha(y)\psi_\alpha(\phi)$ . In this way we get for the  $\chi_\alpha(y)$  and  $\psi_\alpha(\phi)$  parts the following equations

$$\hbar^2 \partial_y^2 \chi_\alpha(y) + (\alpha^2 - 3\lambda_s^2 \kappa e^{-2y/\sqrt{3}}) \chi_\alpha(y) = 0, \quad (21)$$

$$\hbar^2 \partial_\phi^2 \psi_\alpha(\phi) + \left( \alpha^2 - \frac{\lambda_s^2 q^2}{4} e^{-2\phi} \right) \psi_\alpha(\phi) = 0 \quad (22)$$

where  $\alpha$  is the separation constant.

We consider the case for  $\kappa > 0$ , for which the general solutions to both parts are given by

$$\chi_\alpha(y) = A_1 I_{i\frac{\sqrt{3}\alpha}{\hbar}}(3\lambda_s \sqrt{\kappa} e^{-\frac{y}{\sqrt{3}}}/\hbar) + A_2 K_{i\frac{\sqrt{3}\alpha}{\hbar}}(3\lambda_s \sqrt{\kappa} e^{-\frac{y}{\sqrt{3}}}/\hbar), \quad (23)$$

$$\psi_\alpha(\phi) = B_1 I_{i\frac{\alpha}{\hbar}}(\lambda_s q e^{-\phi}/2\hbar) + B_2 K_{i\frac{\alpha}{\hbar}}(\lambda_s q e^{-\phi}/2\hbar). \quad (24)$$

To avoid an infinite value of the wave function as  $y \rightarrow -\infty$  and  $\phi \rightarrow -\infty$  we must choose  $\chi_\alpha(y) \sim K_{i\frac{\sqrt{3}\alpha}{\hbar}}(3\lambda_s \sqrt{\kappa} e^{-\frac{y}{\sqrt{3}}}/\hbar)$  and  $\psi_\alpha(\phi) \sim K_{i\frac{\alpha}{\hbar}}(\lambda_s q e^{-\phi}/2\hbar)$ , which corresponds to the pre-big-bang regime for  $\alpha^2 < 3\lambda_s^2 \kappa e^{-2y/\sqrt{3}}$  and where  $K_{i\nu}(x)$  denotes the MacDonald function of imaginary order. Using the result given in [10] we normalize the  $\chi_\alpha(y)$  part of the wave function in the following form

$$\int_{-\infty}^{\infty} dy \chi_\alpha^*(y) \chi_{\alpha'}(y) = \delta(\alpha^2 - \alpha'^2), \quad (25)$$

and a similar expression can be found for the  $\phi$  part.

In order to write the WDWM equation (17) it is very useful to employ the next relation

$$f(x, p) \star g(x, p) = f\left(x + \frac{i\hbar \vec{\partial}}{2} p, p - \frac{i\hbar \vec{\partial}}{2} x\right) g(x, p), \quad (26)$$

from which we obtain the following equation

$$\left( -\left( P_y - \frac{i\hbar \vec{\partial}}{2} \right)^2 + \left( P_\phi - \frac{i\hbar \vec{\partial}}{2} \right)^2 + \frac{1}{4} \lambda_s^2 q^2 e^{-2\left(\phi + \frac{i\hbar}{2} \vec{\partial}_{P_\phi}\right)} - 3\lambda_s^2 \kappa e^{-\frac{2}{\sqrt{3}}\left(y + \frac{i\hbar}{2} \vec{\partial}_{P_y}\right)} \right) \rho(y, P_y, \phi, P_\phi) = 0. \quad (27)$$

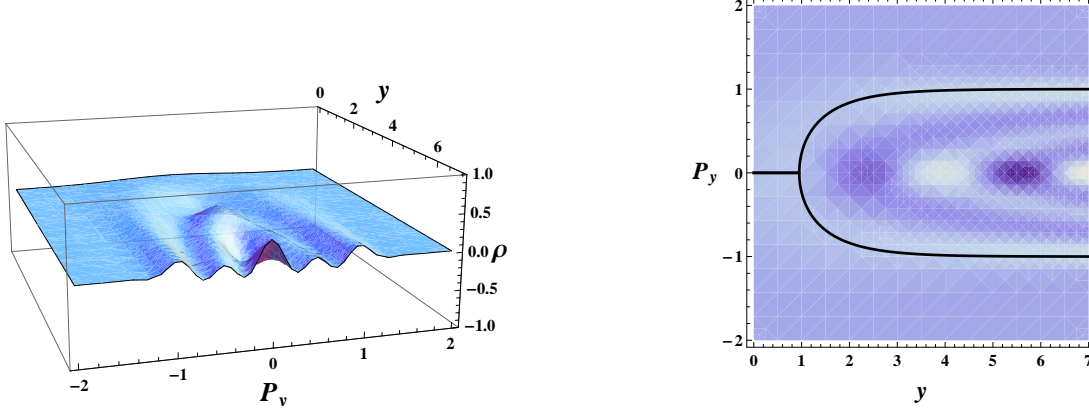


FIG. 1: The Wigner function and its density are plotted in the  $y$  variable ( $\hbar = 1, \alpha = 1$ ). The figure on the left shows many oscillations due to the interference among wave functions of expanding and contracting universes. The thick curve on the right figure corresponds to the classical trajectory which coincides with the maximum of the Wigner function.

The last expression can be split into two equations corresponding to its real part

$$\left[ -P_y^2 + \frac{\hbar^2}{4} \partial_y^2 + P_\phi^2 - \frac{\hbar^2}{4} \partial_\phi^2 + \frac{1}{4} \lambda_s^2 q^2 e^{-2\phi} \cos(\hbar \partial_{P_\phi}) - 3\lambda_s^2 \kappa e^{-\frac{2}{\sqrt{3}}y} \cos\left(\frac{\hbar}{\sqrt{3}} \partial_{P_y}\right) \right] \rho(y, P_y, \phi, P_\phi) = 0, \quad (28)$$

and its imaginary part

$$\left[ \hbar(P_y \partial_y) - \hbar(P_\phi \partial_\phi) - \frac{1}{4} \lambda_s^2 q^2 e^{-2\phi} \sin(\hbar \partial_{P_\phi}) + 3\lambda_s^2 \kappa e^{-\frac{2}{\sqrt{3}}y} \sin\left(\frac{\hbar}{\sqrt{3}} \partial_{P_y}\right) \right] \rho(y, P_y, \phi, P_\phi) = 0. \quad (29)$$

We propose  $\rho(y, P_y, \phi, P_\phi) = \rho_y(y, P_y) \rho_\phi(\phi, P_\phi)$ , and taking into account that  $e^{ia\partial_x} f(x) = f(x + ia)$  then from the two previous equations we obtain the following results:

For the function  $\rho_y(y, P_y)$

$$\begin{aligned} & \left[ -P_y^2 + \mu^2 + \frac{\nu^4}{4P_y^2} \right] \rho_y(y, P_y) + \frac{\hbar}{4P_y} \left\{ \left( \frac{\nu^2 u(y)}{i\hbar P_y} - \frac{2iu(y)}{\sqrt{3}} \right) \left( \rho_y\left(y, P_y + \frac{i\hbar}{\sqrt{3}}\right) - \rho_y\left(y, P_y - \frac{i\hbar}{\sqrt{3}}\right) \right) \right. \\ & - \frac{u^2(y)}{\hbar} \left[ \frac{1}{P_y + \frac{i\hbar}{\sqrt{3}}} \left( \rho_y\left(y, P_y + \frac{2i\hbar}{\sqrt{3}}\right) - \rho_y(y, P_y) \right) - \frac{1}{P_y - \frac{i\hbar}{\sqrt{3}}} \left( \rho_y(y, P_y) - \rho_y\left(y, P_y - \frac{2i\hbar}{\sqrt{3}}\right) \right) \right] \\ & \left. + \frac{\nu^2 u(y)}{i\hbar} \left[ \frac{\rho_y(y, P_y + \frac{i\hbar}{\sqrt{3}})}{P_y + \frac{i\hbar}{\sqrt{3}}} - \frac{\rho_y(y, P_y - \frac{i\hbar}{\sqrt{3}})}{P_y - \frac{i\hbar}{\sqrt{3}}} \right] \right\} - u(y) \left( \rho_y\left(y, P_y + \frac{i\hbar}{\sqrt{3}}\right) + \rho_y\left(y, P_y - \frac{i\hbar}{\sqrt{3}}\right) \right) = 0, \quad (30) \end{aligned}$$



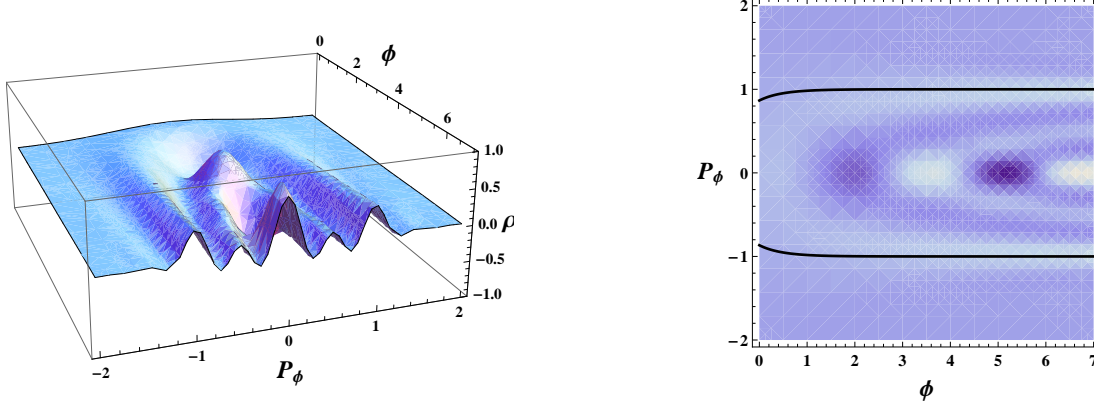


FIG. 2: The Wigner function and its density are plotted in the  $\phi$  variable ( $\hbar = 1, \alpha = 1$ ). The figure on the left shows oscillations of higher amplitude with respect to the  $y$  part. In the right figure the thick curve is the classical trajectory and corresponds the maximum of the Wigner function (the  $\phi < 0$  region is classical forbidden).

and for the function  $\rho_\phi(\phi, P_\phi)$

$$\begin{aligned} & \left[ P_\phi^2 - \mu^2 - \frac{\nu^4}{4P_\phi^2} \right] \rho_\phi(\phi, P_\phi) - \frac{\hbar}{4P_\phi} \left\{ \left( \frac{\nu^2 w(\phi)}{i\hbar P_\phi} - 2iw(\phi) \right) (\rho_\phi(\phi, P_\phi + i\hbar) - \rho_\phi(\phi, P_\phi - i\hbar)) \right. \\ & - \frac{w^2(\phi)}{\hbar} \left[ \frac{1}{P_\phi + i\hbar} (\rho_\phi(\phi, P_\phi + 2i\hbar) - \rho_\phi(\phi, P_\phi)) - \frac{1}{P_\phi - i\hbar} (\rho_\phi(\phi, P_\phi) - \rho_\phi(\phi, P_\phi - 2i\hbar)) \right] \\ & \left. + \frac{\nu^2 w(\phi)}{i\hbar} \left[ \frac{\rho_\phi(\phi, P_\phi + i\hbar)}{P_\phi + i\hbar} - \frac{\rho_\phi(\phi, P_\phi - i\hbar)}{P_\phi - i\hbar} \right] \right\} + w(\phi) (\rho_\phi(\phi, P_\phi + i\hbar) + \rho_\phi(\phi, P_\phi - i\hbar)) = 0, \end{aligned} \quad (31)$$

where we have defined the functions  $u(y) = \frac{3\lambda_s^2 \kappa e^{-\frac{2}{\sqrt{3}}y}}{2}$  and  $w(\phi) = \frac{\lambda_s^2 q^2 e^{-2\phi}}{8}$ .

It is hard to solve the last two equations, so to obtain their solutions we will follow a different approach and will use the integral representation for the Wigner function. We can find its  $y$  part by computing

$$\begin{aligned} \rho_y(y, P_y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^*(y - \hbar q/2) \exp(-iqP_y) \psi(y + \hbar q/2) dq \\ &= \frac{|A_2|^2}{2\pi} \int_{-\infty}^{\infty} K_{i\frac{\sqrt{3}\alpha}{\hbar}}^* (3\lambda_s \sqrt{\kappa} e^{-\frac{(y-\hbar q/2)}{\sqrt{3}}}/\hbar) \exp(-iqP_y) K_{i\frac{\sqrt{3}\alpha}{\hbar}} (3\lambda_s \sqrt{\kappa} e^{-\frac{(y+\hbar q/2)}{\sqrt{3}}}/\hbar) dq. \end{aligned} \quad (32)$$

Defining the variables  $\omega = e^{\hbar q/2\sqrt{3}}$  and  $z = a = 3\frac{\lambda_s}{\hbar} \sqrt{\kappa} e^{-y/\sqrt{3}}$ , we get:

$$\rho_y(y, P_y) = \frac{\sqrt{3}|A_2|^2}{\pi\hbar} \int_0^\infty K_{i\frac{\sqrt{3}\alpha}{\hbar}}(z\omega) \omega^{\sigma-\frac{1}{2}} K_{i\frac{\sqrt{3}\alpha}{\hbar}}(z/\omega) d\omega, \quad (33)$$

where  $\sigma = -\frac{1}{2} - \frac{2\sqrt{3}}{\hbar} iP_y$  and  $|A_2|^2 = \frac{\sqrt{3} \sinh(\frac{\pi\sqrt{3}\alpha}{\hbar})}{\pi^2 \hbar^2}$ .

Using now the following result (see Sec. 19.6 formula (25) in [11] and the comment in [12])

$$\int_0^\infty dw (wz)^{1/2} w^{\sigma-1} K_\mu(a/w) K_\nu(wz) = 2^{-\sigma-5/2} a^\sigma G_{04}^{40} \left( \frac{a^2 z^2}{16} \middle| \frac{\mu-\sigma}{2}, \frac{-\mu-\sigma}{2}, \frac{1}{4} + \frac{\nu}{2}, \frac{1}{4} - \frac{\nu}{2} \right), \quad (34)$$

where  $G_{04}^{40} \left( \frac{a^2 z^2}{16} \middle| \frac{\mu-\sigma}{2}, \frac{-\mu-\sigma}{2}, \frac{1}{4} + \frac{\nu}{2}, \frac{1}{4} - \frac{\nu}{2} \right)$  is a special case of Meijer's  $G$  function (see Sec. 5.3 in [13])

$$G_{pq}^{mn} \left( z \middle| \begin{matrix} a_i, & i = 1, \dots, p \\ b_j, & j = 1, \dots, q \end{matrix} \right), \quad (35)$$

we obtain the following expression

$$\begin{aligned} \rho_y(y, P_y) &= \frac{\sinh(\frac{\pi\sqrt{3}\alpha}{\hbar})}{4\pi^3\hbar^2} \frac{e^{\frac{y}{\sqrt{3}}}}{\lambda_s\sqrt{\kappa}} \left( \frac{3u(y)}{2\hbar^2} \right)^{-\frac{\sqrt{3}}{\hbar}iP_y} \\ &\times G_{04}^{40} \left( \frac{9u^2(y)}{4\hbar^4} \middle| \frac{1}{4} + \frac{i\sqrt{3}}{\hbar} \left( \frac{\alpha}{2} + P_y \right), \frac{1}{4} + \frac{i\sqrt{3}}{\hbar} \left( \frac{-\alpha}{2} + P_y \right), \frac{1}{4} + \frac{i\sqrt{3}\alpha}{2\hbar}, \frac{1}{4} - \frac{i\sqrt{3}\alpha}{2\hbar} \right). \end{aligned} \quad (36)$$

Now, employing the Meijer's function property

$$x^\sigma G_{pq}^{mn} \left( x \middle| \begin{matrix} a_i, & i = 1, \dots, p \\ b_j, & j = 1, \dots, q \end{matrix} \right) = G_{pq}^{mn} \left( x \middle| \begin{matrix} a_i + \sigma, & i = 1, \dots, p \\ b_j + \sigma, & j = 1, \dots, q \end{matrix} \right), \quad (37)$$

equation (36) can be written as

$$\begin{aligned} \rho_y(y, P_y) &= \frac{\sinh(\frac{\pi\sqrt{3}\alpha}{\hbar})}{4\pi^3\hbar^2} \frac{e^{\frac{y}{\sqrt{3}}}}{\lambda_s\sqrt{\kappa}} \\ &\times G_{04}^{40} \left( \frac{9u^2(y)}{4\hbar^4} \middle| \frac{1}{4} + \frac{i\sqrt{3}}{2\hbar}(\alpha + P_y), \frac{1}{4} + \frac{i\sqrt{3}}{2\hbar}(-\alpha + P_y), \frac{1}{4} + \frac{i\sqrt{3}}{2\hbar}(\alpha - P_y), \frac{1}{4} + \frac{i\sqrt{3}}{2\hbar}(-\alpha - P_y) \right). \end{aligned} \quad (38)$$

It is possible to verify that this Wigner function indeed satisfy equation (30).

Performing a similar procedure we can obtain the following Wigner function for the  $\phi$  part

$$\begin{aligned} \rho_\phi(\phi, P_\phi) &= \frac{\sinh(\frac{\pi\alpha}{\hbar})}{2\pi^3\hbar^2} \frac{e^\phi}{\lambda_s q} \\ &\times G_{04}^{40} \left( \frac{w^2(\phi)}{4\hbar^4} \middle| \frac{1}{4} + \frac{i}{2\hbar}(\alpha + P_\phi), \frac{1}{4} + \frac{i}{2\hbar}(-\alpha + P_\phi), \frac{1}{4} + \frac{i}{2\hbar}(\alpha - P_\phi), \frac{1}{4} + \frac{i}{2\hbar}(-\alpha - P_\phi) \right), \end{aligned} \quad (39)$$

which fulfills Eqn. (31).

We can gain some physical insight if we plot the Wigner functions of the corresponding  $y$  and dilaton parts for several values of  $\alpha$ . For the  $y$  part and  $\alpha = 1$  Fig. 1 shows that the classical trajectory is near the highest peaks of the Wigner function. For values of  $\alpha$  smaller than one there are less oscillations but the classical trajectory does not correspond to the highest peaks, in fact, there is an ample region where the Wigner function is large. For

values of  $\alpha$  bigger than one it can be observed an increment in the number of oscillations of Wigner function and the peaks of the oscillations are far away from the classical trajectory (these plots are not showed in the paper). We conclude that the quantum interference effects are enhanced for larger values of  $\alpha$ . A similar behavior is obtained for the dilaton part for  $\alpha$ , nevertheless from Fig. 2 its amplitude is larger. The parameter  $\alpha$  can be interpreted as the energy of the  $y$  and  $\phi$  parts, the matter (dilaton axion part) has positive energy and any fluctuation of it induces a variation of the energy in the gravitational part (the scale factor) in order to vanish the total energy.

#### IV. FINAL REMARKS

In this paper we have presented the WWGM formalism for a gravitational field coupled to matter in the flat superspace (and flat phase superspace) where the Stratonovich-Weyl quantizer, the star product and the Wigner functional are obtained. These results can be used in general situations but in a first approach we applied [1] the formalism to some interesting minisuperspace models widely studied in the literature, in particular we studied here the axion-dilaton quantum cosmology model with non-vanishing spatial curvature. We found the corresponding WDWM equation and the equivalent differential-difference equation. These equations have an exact solution in terms of the Meijer's functions. We have seen that the parameter  $\alpha$  has an interpretation of energy for the components  $y$  and  $\phi$ . For an arbitrary total constant energy any fluctuation in the energy of matter does induce a corresponding fluctuation in the gravitational part in such a way that both are compensated. Moreover, we found that for small values of  $\alpha$  there are fewer oscillations in the gravitational part of the Wigner function but the classical trajectory does not correspond to the highest peaks. In the case of bigger values of  $\alpha$  there is a clear increment in the number of oscillations of Wigner function, being its peaks far away from the classical trajectory. We observe also that the quantum interference effects for larger values of  $\alpha$  are enhanced. The same situation is regarded for the dilaton part but with larger amplitude oscillations. It is worth to mention that the construction presented here can be employed to treat other cosmological models.

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